TEMPERATURE FIELD OF ROTATING ANNULAR DISKS

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The axially asymmetric problem of the temperature of a rotating disk is examined; the solution is dependent on polar coordinates and time. The steady temperature field in the disk following transient processes is analyzed.

The machining of wood, fiber boards and plastics, composition materials, precious stones, and a number of other materials by cutting is accomplished by an instrument, the body (substance) of which is a flexible disk 0.5-5.5 mm thick. The exact operation of this instrument depends on its dynamic stability, which is connected for the most part with the presence of unfavorable thermal stresses in the periphery of the disk. The unfavorable temperature stresses arise as a consequence of unequal heating of the disk along a radius caused by location of the heat source in a restricted area of the disk periphery and the intensity of its cooling through heat exchange with the surrounding medium. In order to determine the temperature stresses one must know the distribution law of the temperature stresses along the radius and along concentric circles of the body of the instrument. The temperature distribution along the radius of a disk for the conditions of an axially symmetric problem with a steady temperature system, where one assumes the condition that the heat acts equally along the entire peripheral cylindrical surface of the disk, is examined in [1]. In this case an annular, rather than a solid disk is examined, for which the boundary condition $T(r_0, \varphi, t) = T_0$ is assumed, in connection with which its thermal resistance in the radial direction is calculated to be close to 1100 times greater than in the axial direction. The correctness of the given boundary condition was confirmed experimentally in [2], according to the data of which even for the part of a disk with a radius of 0.4-0.5 of the outer radius, the temperature is equal to the surrounding air temperature T_0.

Heating of a rotating disk by a point source of heat was examined in [3]. However in [3] a stationary temperature field was considered, but transient processes were not reflected, although they do have a considerable interest for a number of practical cases. In the first place this involves the formation of local temperature stresses in the disk, sharp temperature variation in the cutting area of the disk, which leads to alteration in the metal structure, etc.

In the present work the temperature field is determined for a rotating circular disk, on the outer rim of which slides a point heat source. The temperature of the inner part of the disk is taken as constant and equal to the temperature of the surrounding medium.

The equation of thermal conductivity in our case has the form

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial\varphi^2}\right)T(r,\varphi,t)-\frac{2h}{b\lambda}\left[T(r,\varphi,t)-T_0\right]+\frac{1}{\lambda}Q(r,\varphi,t)=\frac{1}{\alpha}\cdot\frac{\partial T(r,\varphi,t)}{\partial t}. \tag{1}$$

The solution should satisfy the initial and boundary conditions

$$T(r,\varphi,0)=T_0, \tag{2}$$

$$T(r_0,\varphi,t)=T_0, \tag{3}$$

and also the condition of periodicity in φ

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$$T(r, \varphi, t) = T(r, \varphi + 2k\pi, t) \quad (k = 1, 2, \ldots). \tag{4}$$

Changing to a new variable

$$u(r, \varphi, t) = T(r, \varphi, t) - T_0, \tag{5}$$

we seek a solution in the form

$$u(r, \varphi, t) = \sum_{m=0}^{\infty} u_m(t) J_m\left(\mu_m \frac{r}{R}\right) \cos m\varphi. \tag{6}$$

The periodicity in φ is satisfied for integers m, while the numbers μ_m are determined from the boundary condition (3), from which it follows that

$$J_m\left(\mu_m \frac{r_0}{R}\right) = 0, \tag{7}$$

where $\mu_m r_0/R$ is the (m + 1)-th root of this equation. We substitute (6) and the analogous expansion of $Q(r, \varphi, t)$ in Eq. (1):

$$\frac{du_m(t)}{dt} + \alpha\left[\left(\frac{\mu_m}{R}\right)^2 + \frac{2h}{b\lambda}\right] u_m(t) = \frac{\alpha}{\lambda} Q_m(t). \tag{8}$$

The solution of (8), satisfying the initial condition $u_m(0) = 0$, takes the form

$$u_m(t) = \frac{\alpha}{\lambda} \int_0^t Q_m(t') \exp\left[-\psi_m(t - t')\right] dt', \tag{9}$$

where

$$\psi_m = \alpha\left[\left(\frac{\mu_m}{R}\right)^2 + \frac{2h}{b\lambda}\right] \quad (m = 0, 1, 2, \ldots). \tag{10}$$

Determining the expansion coefficients $Q_m(t)$ and substituting them in (9) we obtain for $u(r, \varphi, t)$ the expression

$$\begin{aligned} u(r, \varphi, t) = \frac{\alpha}{\pi\lambda} \int_0^t dt' \Big\{ &\Omega_0 J_0(\mu_0 x) \exp\left[-\psi_0(t - t')\right] \\ &\times \int_0^{2\pi} d\varphi' \int_{x_0}^1 Q(x', \varphi', t') J_0(\mu_0 x') x' dx' \\ &+ 2\sum_{m=1}^{\infty} \Omega_m J_m(\mu_m x) \cos m\varphi \exp\left[-\psi_m(t - t')\right] \\ &\times \int_0^{2\pi} \cos m\varphi' d\varphi' \int_{x_0}^1 Q(x', \varphi', t') J_m(\mu_m x') x' dx' \Big\}. \end{aligned} \tag{11}$$

Here $x = r/R$; $x_0 = r_0/R$, and

$$\Omega_m = \{[J_m(\mu_m)]^2 - J_{m-1}(\mu_m) J_{m+1}(\mu_m)\}^{-1} \quad (m = 0, 1, 2, \ldots). \tag{12}$$

To evaluate the integrals entering into (11) we use the method presented in [4]. We replace the (discrete) point source of heat moving continuously around the periphery of the disk by a point source moving discontinuously from point to point

$$Q(x, \varphi, t) = \frac{\delta(x - 1)}{R} \sum_{i=1}^{I} Q_i \delta(\varphi - \varphi_i) \delta(t - t_i). \tag{13}$$

Here i is the ordinal number of the discontinuous source, I is the total number of these discontinuities, and

$$Q_i = Q/s\nu n_0,$$

where n_0 is the number of revolutions per second of the disk, ν is the number of discrete sources included in one period, and $s = 2\pi Rb$.

From evaluation of the integrals with respect to x', φ', and t' we obtain in place of (11)

$$u(r,\ \varphi,\ t) = \frac{\alpha}{\pi\lambda R} \sum_{i=1}^{I} Q_i \left\{ \Omega_0 J_0(\mu_0) J_0\left(\mu_0 \frac{r}{R}\right) \exp\left[-\psi_0 (t - t_i)\right] \right.$$

$$\left. + 2 \sum_{m=1}^{\infty} \Omega_m J_m(\mu_m) J_m\left(\mu_m \frac{r}{R}\right) \cos m\varphi \cos m\varphi_i \exp\left[-\psi_m (t - t_i)\right] \right\}. \tag{14}$$

We further substitute φ_i and $t-t_i$ in the form

$$\varphi_i = \frac{2\pi}{\nu} i, \quad t - i_i = \frac{i-1}{\nu n_0} \tag{15}$$

and total with respect to i the sums in (14). We then replace the discontinuous source with the continuously moving point source, for which we determine the limit of the resulting expression at $\nu \to \infty$. Finally we obtain the solution of Eq. (1) in the form

$$T(r,\ \varphi,\ t) = T_0 + \frac{Q}{2\pi^2 b\lambda} \left\{ A_0 J_0\left(\mu_0 \frac{r}{R}\right) \left[1 - \exp(-\psi_0 t)\right] \right.$$

$$\left. + 2 \sum_{m=1}^{\infty} A_m J_m \left(\mu_m \frac{r}{R}\right) \cos m\varphi \left[1 - \exp(-\psi_m t) \left(\cos m\omega t - \frac{m\omega}{\psi_m} \sin m\omega t\right)\right] \right\}. \tag{16}$$

Here

$$A_m = \frac{\alpha \Omega_m J_m(\mu_m) \psi_m}{R^2 (\psi_m^2 + m^2\omega^2)} \quad (m = 0,\ 1,\ 2,\ \ldots). \tag{17}$$

We introduce the dimensionless temperature

$$\tau = \frac{T(r,\ \varphi,\ t) - T_0}{Q/(2\pi^2 b\lambda)}. \tag{18}$$

Equation (16) is transformed to the form

$$\tau = A_0 J_0 \left(\mu_0 \frac{r}{R}\right) \left[1 - \exp(-\psi_0 t)\right]$$

$$+ 2 \sum_{m=1}^{\infty} A_m J_m \left(\mu_m \frac{r}{R}\right) \cos m\varphi \left[1 - \exp(-\psi_m t) \right.$$

$$\left. \times \left(\cos m\omega t - \frac{m\omega}{\psi_m} \sin m\omega t\right)\right]. \tag{19}$$

Equation (19) describes both the transient process and the steady temperature field arising after its completion.

We now examine the case in which one may ignore the transient processes. We discard the parts in (19) which diminish with time. Furthermore we introduce the following substitution:

$$\varphi = \varphi_0 + \omega t, \tag{20}$$

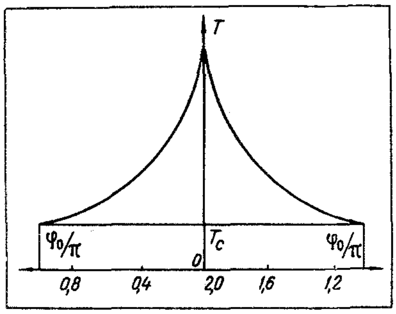

Fig. 1

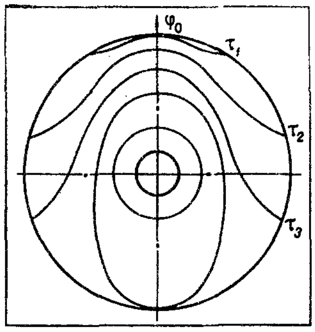

Fig. 2

Fig. 1. Dependence of dimensionless temperature on the angular coordinate of an arbitrary chosen point of a stationary disk.

Fig. 2. Diagram of isotherms in a stationary disk. $\tau_1 > \tau_2 > \tau_3$ (τ is the dimensionless temperature).

where φ_0 is the angular coordinate of an arbitrary point of the disk stationary with respect to the source. Then we obtain from (19)

$$\tau = A_0 J_0\left(\mu_0 \frac{r}{R}\right) + 2\sum_{m=1}^{\infty} A_m J_m\left(\mu_m \frac{r}{R}\right)\cos m(\varphi_0 + \omega t). \tag{21}$$

The result obtained may be interpreted in the following way. The stationary point heat source is located at the point with polar coordinates (R, 0). A stationary temperature field is established in planar coordinates satisfying the boundary conditions. The temperature at the arbitrary point of the rotating disk undergoes harmonic oscillation, taking the periodic value τ, unambiguously determined by the coordinates of the corresponding point of the disk. The greatest amplitude of oscillation occurs at the peripheral points whose temperature varies within the limits from $\tau(R, 0)$ to $\tau(R, \pi)$ (highest and lowest values, respectively). The amplitude of oscillation decreases with a decrease in r and at points with $r = r_0$ the temperature is constant and equals $\tau_0 = 0$.

For the case of a stationary disk ($\omega = 0$) we have

$$\tau = A_0 J_0\left(\mu_0 \frac{r}{R}\right) + 2\sum_{m=1}^{\infty} A_m J_m\left(\mu_m \frac{r}{R}\right)\cos m\varphi_0. \tag{22}$$

The dependence of temperature on φ_0 and the isotherm in a stationary disk are shown in Figs. 1 and 2.

In the case of very rapid rotation of the disk ($\omega \to \infty$) we obtain from Eq. (21)

$$\tau = A_0 J_0\left(\mu_0 \frac{r}{R}\right). \tag{23}$$

The temperature field in this case is axially symmetrical since a constant temperature is eatablished at the periphery of the disk (the heat source is found simultaneously at all points of the outer rim of the disk). The isotherms take the form of concentric circles.

NOTATION

r and φ	are polar coordinates;
t	is the time;
r_0 and R	are the inner and outer radii of the disk;
b	is the disk thickness;
λ	is the thermal conductivity coefficient;
h	is the heat exchange coefficient;
α	is the thermal diffusion coefficient;
$Q(r, \varphi, t)$	is the heat generated per unit time per unit volume of the disk;

$T(r, \varphi, t)$	is the temperature at an arbitrary point of the disk;
T_0	is the temperature of the surrounding medium;
ω	is the angular velocity of the disk;
J_m	is a Bessel function of the first kind of the order m;
$\delta(x)$	is the delta function.

LITERATURE CITED

1. A. É. Grube, V. I. Sanev, and V. K. Pashkov, Lesnoi Zhurnal, No. 3 (1966).
2. A. É. Grube, V. I. Sanev, and V. K. Pashkov, ibid., No. 4 (1969).
3. E. G. Hauptmann and H. Ramsey, Appl. Sci. Res., 20, 436 (1969).
4. M H. Gobble, Intern. J. Heat Mass Transfer, 10, 1281 (1967).